

Flows of Dense Granular Media

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Abstract

We review flows of dense cohesionless granular materials, with a special focus on the question of constitutive equations. We first discuss the existence of a dense flow regime characterized by enduring contacts. We then emphasize that dimensional analysis strongly constrains the relation between stresses and shear rates, and show that results from experiments and simulations in different configurations support a description in terms of a frictional visco-plastic constitutive law. We then discuss the successes and limitations of this empirical rheology in light of recent alternative theoretical approaches. Finally, we briefly present depth-averaged methods developed for free surface granular flows.

1. INTRODUCTION

Those who have played with dry sand on the beach or with sugar in their kitchen are aware that a collection of solid grains can behave macroscopically like a liquid and flow. However, understanding and modeling this common observation are difficult tasks, and granular flows have been the subject of intense activity at the leading edge of fluid mechanics, soil mechanics, rheology, and statistical physics. Research is motivated by numerous applications encountered in industrial processes and especially in geophysics for the description and prediction of natural hazards such as landslides, rock avalanches, and pyroclastic flows. However, the recent interest in granular flows certainly also is stimulated by new fundamental questions raised by this peculiar fluid, which shares similarities with other athermal disordered systems such as foam, amorphous solids, or emulsions (Liu & Nagel 1998) and exhibits a rich phenomenology (Aranson & Tsimring 2006).

Among these important questions is the lack of constitutive equations. Whereas classical fluids are well described by the Navier-Stokes equations, no constitutive law can reproduce the diversity of behavior observed with a cohesionless granular material. This difficulty originates from fundamental characteristics of granular matter such as negligible thermal fluctuations, highly dissipative interactions, and a lack of separation between the microscopic grain scale and the macroscopic scale of the flow (Goldhirsch 2003). As a result, granular flows are often classified into three different regimes (Jaeger et al. 1996): a dense quasi-static regime in which the deformations are very slow and the particles interact by frictional contacts (Roux & Combes 2002); a gaseous regime in which the flow is very rapid and dilute, and the particles interact by collision (Goldhirsch 2003); and an intermediate liquid regime in which the material is dense but still flows like a liquid, the particles interacting both by collision and friction (GDR MiDi 2004, Pouliquen & Chevoir 2002).

In this article, we focus on this last liquid regime, which is most often encountered in applications, and discuss the possibility of a hydrodynamic description of dense granular flows. We restrict the review to rigid dry grains and do not consider soft particles, cohesive effects, or interaction with a surrounding fluid. In Section 2, we explore the different flow regimes and discuss the granular solid-liquid and liquid-gas transition. In Section 3, we focus on the rheological properties of the granular liquid. We show that dimensional arguments suggest a simple frictional rheology, providing a hydrodynamic framework that gives quantitative predictions in some simple configurations. However, serious limitations to this view exist, which are discussed in light of recently developed alternative theoretical approaches. In Section 4, we present depth-averaged approaches for granular flows, which represent an alternative strategy to circumvent the lack of constitutive equations capable of capturing the whole phenomenology of granular flows.

2. DIFFERENT FLOW REGIMES

Figure 1 shows a typical granular flow obtained when pouring beads on a pile. One clearly observes three different regions corresponding to the three different flow

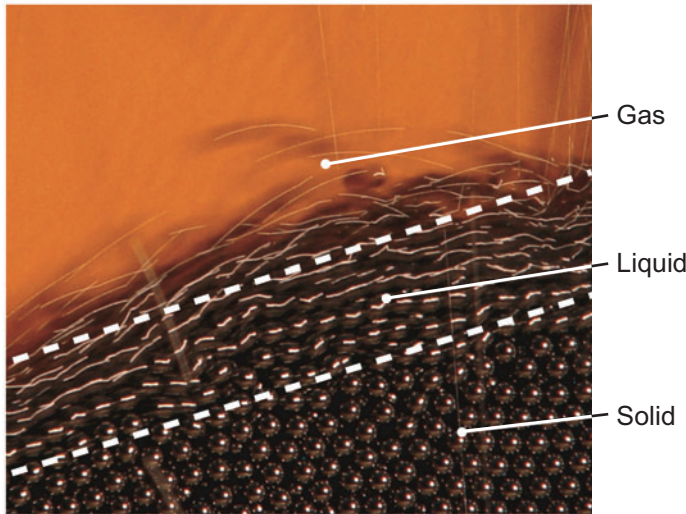


Figure 1

An illustration of the solid, liquid, and gas flow regimes obtained by pouring steel beads on a pile.

regimes: a solid region under the pile in which grains do not move or creep very slowly, a liquid region in which a dense layer flows, and a gaseous region in which the beads bounce in all directions creating a dilute chaotic medium. In the following, we discuss the boundaries between the different flow regimes.

2.1. Transition Between Solid and Liquid Behaviors

Our daily experience tells us that a pile of sand has to be inclined above a critical angle in order to flow. This is because the onset of the flow of granular materials is given by a friction criterion: The ratio of shear stress to normal stress, which is simply the tangent of the slope of the pile, has to reach a critical value called the friction coefficient in order for the material to deform. The reason why the solid-liquid transition for a granular material is a friction criterion is that, for rigid grains, no internal stress scale exists. This contrasts with other complex fluids exhibiting flow threshold such as Bingham fluids, in which an internal stress scale exists that is linked to the breakage of a microscopic structure. From a microscopic point of view, the strength of a granular material comes not only from the friction between grains, but also from the entanglement of the particles: Packed frictionless particles still exhibit a macroscopic friction coefficient.

Although a friction criterion is the zero-order description of the solid-liquid transition, the details are more complex. First, the initiation of the flow is sensitive to the initial preparation of the sample and depends on both the initial volume fraction and the history of previous deformations (Daerr & Douady 1999a). Modeling the initial deformation and the coupling between strain, stress, volume fraction, and possibly other texture fields, such as contact orientation, is the domain of soils mechanics (Roux & Radjai 1998, Schofield & Wroth 1968, Wood 1990). Researchers have proposed plastic constitutive laws, and attempts to connect the microstructure to

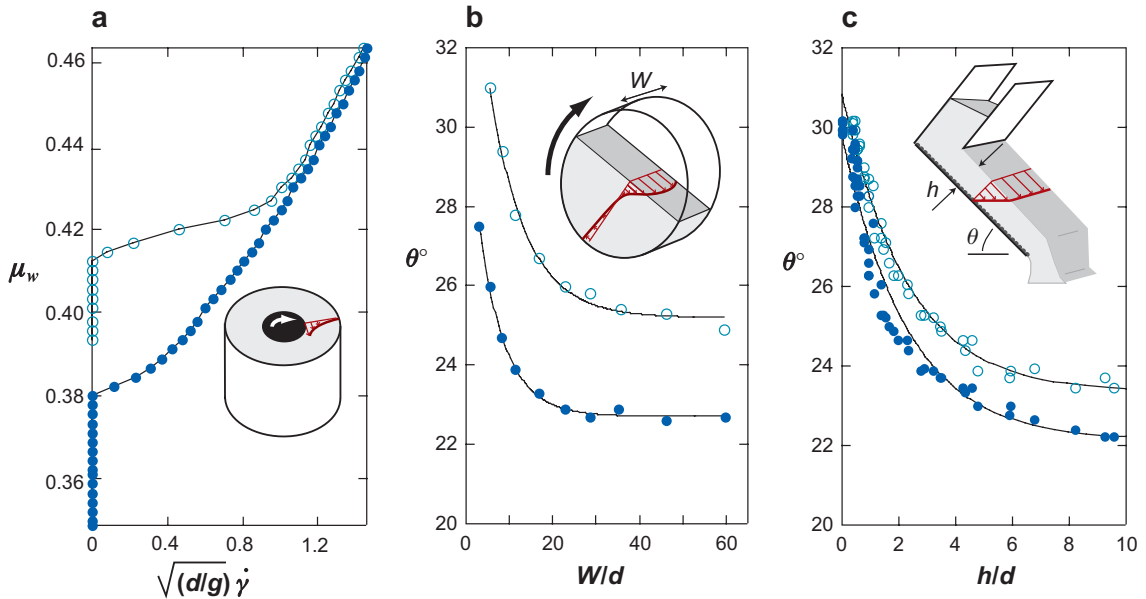


Figure 2

Hysteresis and flow threshold in different systems. (a) Cylindrical Couette cell, with the friction coefficient at the inner wall plotted as a function of the dimensionless mean shear rate. Open circles represent increasing shear stress, and filled circles represent decreasing shear stress (da Cruz et al. 2002). (b) Rotating drum with θ_{start} (open circles) and θ_{stop} (filled circles) plotted as functions of the width of the drum (Courrech du Pont et al. 2003). (c) Inclined plane with θ_{start} (open circles) and θ_{stop} (filled circles) plotted as functions of the layer thickness b (Pouliquen & Forterre 2002).

the macroscopic behavior motivate many studies (Roux & Combes 2002). However, such approaches focus on the initiation of the deformation and do not predict what happens when continuous quasi-static flow is imposed on the material (Fenistein et al. 2004, Losert et al. 2000, Mueth et al. 2000, Veje et al. 1999).

A second shortcoming of the ideal friction criterion is that it cannot describe the hysteresis observed in a stress-driven system. In a Couette cell, for example, one has to increase the applied shear stress up to a critical value to induce flow, but once it flows, the material stops only if the shear stress is decreased below a value less than the starting value (**Figure 2a**). For free surface flows, as in a partially filled drum or on a pile, one has to incline the free surface above a critical angle θ_{start} to trigger an avalanche, but the flow will stop only below a lower critical angle θ_{stop} . The origin of hysteresis in granular media is well illustrated by the toy model of a single bead flowing down a rough inclined substrate (Quartier et al. 2000). This analysis clearly shows how hysteresis comes from the balance between kinetic energy, energy dissipation due to collision, and the potential trap made by the roughness of the substrate.

The last weakness of the simple friction criterion is that the flow threshold depends on the system size. In a rotating drum or on a pile, the width of the device plays an

important role, and the two critical angles θ_{start} and θ_{stop} increase when the distance between sidewalls is small (**Figure 2b**). For the case of a granular layer of thickness b on a rough inclined plane (**Figure 2c**), investigators found that the critical angles controlling the start and stop depend on b . Both $\theta_{start}(b)$ and $\theta_{stop}(b)$ increase when the thickness is decreased (Daerr & Douday 1999b, Pouliquen & Forterre 2002, Pouliquen & Renaut 1996). This is the signature of nontrivial finite-size effects and/or boundary effects that are not well understood.

2.2. Transition Between Liquid and Gas Behaviors

At the other extreme, when the granular media is strongly agitated and flows very rapidly, the medium looks like a gas with particles interacting mainly by binary collisions. In this rapid and dilute regime, the analogy with a gas has been fruitful, and much effort has been devoted to the development of a kinetic theory of granular gases (Campbell 1990, Goldhirsch 2003). The difference with classical gases is that kinetic energy is lost during collisions, which is characterized by the restitution coefficient e measuring the ratio between velocities after and before a collision. This approach provides a set of constitutive equations connecting the mean density, the mean velocity, and the so-called granular temperature, which measures the velocity fluctuations (Jenkins & Richman 1985, Jenkins & Savage 1983). This theory has proven to successfully describe the dilute regime (Forterre & Pouliquen 2002, Mitarai & Nakanishi 2004, Swinney & Rericha 2004). However, contrary to optimistic expectations, its range of validity appears to be narrow. When the flow becomes denser, the energy dissipation due to inelasticity becomes so efficient that the basic assumptions of binary collisions and molecular chaos break down: One quickly enters the dense liquid regime.

Researchers have investigated the transition between the liquid and kinetic regimes only in two dimensions by means of the numerical simulation of disks in plane shear (da Cruz et al. 2004, Lois et al. 2006). An important result is that the transition depends on the restitution coefficient e . Different criteria have been proposed to discriminate between liquid and gas regimes based on either the mean duration of contact (da Cruz et al. 2004) or the correlation length of the force network (Lois et al. 2005, 2006), which increases in the liquid regime. Both criterion give qualitatively the same phase diagram in the $\Phi - e$ plane, where Φ is the volume fraction defined as the ratio of the volume occupied by the grains to the total volume (**Figure 3**). In this diagram, there exists a critical volume fraction Φ_c above which continuous flows are not possible and particles have to deform to flow (Campbell 2002, Shen & Sankaran 2004). The range of volume fraction in which the liquid regime is observed increases when the coefficient of restitution decreases. The phase diagram has been established for 2D disks, but one expects a qualitatively similar plot in three dimensions, perhaps a delayed transition to the gaseous state, the dissipation being more efficient in three dimensions.

2.3. The Granular Liquid

In between the solid and gas regime, the granular material flows like a liquid, a regime characterized by enduring contacts between particles and the existence of a

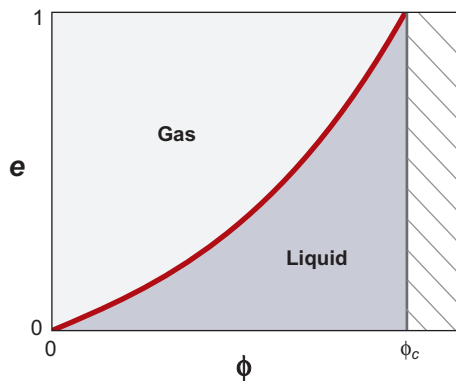


Figure 3

Schematic phase diagram. Φ is the volume fraction, and e is the restitution coefficient.

force network. In this regime, the properties are almost insensitive to the coefficient of restitution e (GDR MiDi 2004), although as discussed above, the transition to the kinetic regime depends on e . To better understand this liquid regime, researchers have investigated different flow configurations, the most common presented in **Figure 4**. These configurations can be divided in two families: flows confined between walls as in shear cells or silo and free surface flows such as flows down an inclined plane, flows in a rotating drum, or flows on a pile. GDR MiDi (2004) discusses in detail the characteristics of these configurations in terms of velocity profiles, density profiles, and velocity fluctuations. Recently, by analogy with classical hydrodynamics problems, more complex flow configurations have been analyzed, such as dam-break problems (Lajeunesse et al. 2004, Lube et al. 2004), coating-like problems (Deboeuf et al. 2006, Felix & Thomas 2004), mixing experiments (Ottino & Khakhar 2000), split Couette devices (Fenistein et al. 2004), drag problems (Hill et al. 2005), and instabilities (Aranson & Tsimring 2006).

A recurrent and central question underlying all the studies involves the constitutive equations of this peculiar liquid. Dense granular flows can be placed in the viscoplastic family of materials because of two broad properties. First, a flow threshold exists, although it is expressed in terms of friction instead of yield stress, as in a classical viscoplastic material. Second, when the material is flowing, shear rate dependence is observed, which gives it a viscous-like behavior. In the following section, we present recent advances in our understanding of the rheology of dense granular flows.

3. RHEOLOGY OF DENSE GRANULAR FLOWS

3.1. Dimensional Analysis: Plane Shear

We first consider the simplest flow configuration consisting of spherical grains of diameter d and density ρ_p sheared between two rough plates at a shear rate $\dot{\gamma}$ in the absence of gravity (**Figure 4a**). A shear stress τ then develops on the top plate. It is

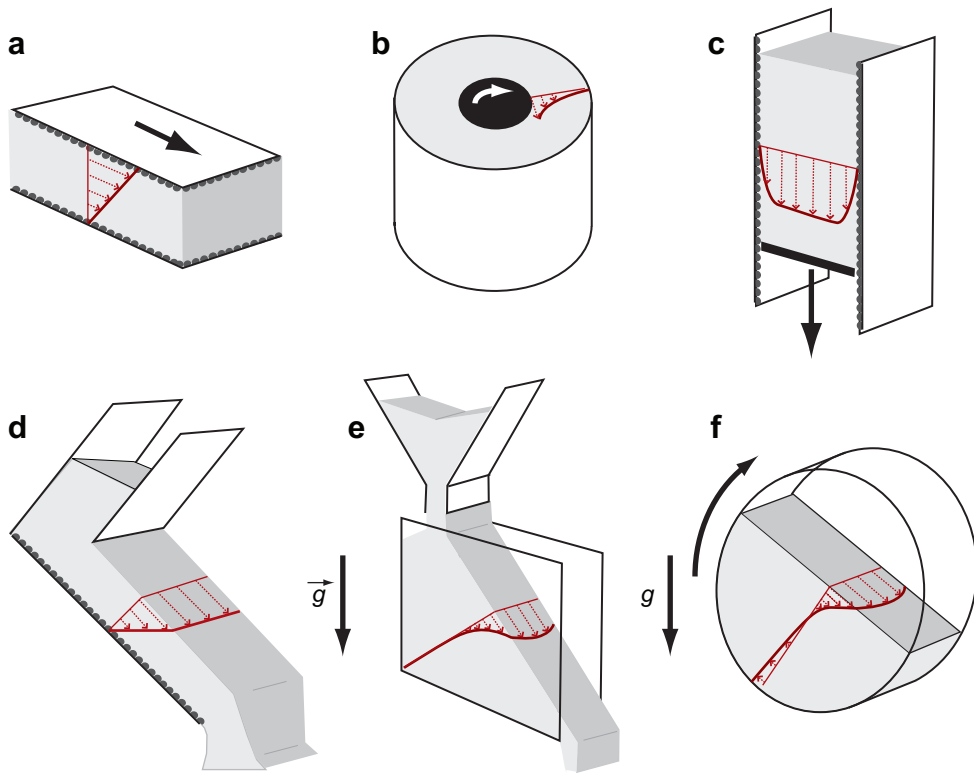


Figure 4

Different flow configurations: (a) plane shear, (b) Couette cell, (c) silo, (d) flows down an inclined plane, (e) flows on a pile, and (f) flows in a rotating drum.

important to notice that there are two ways of shearing the material. The first is to impose the pressure P on the top plate. In this case the upper plate is free to move vertically, and the volume fraction Φ typically decreases with increasing shear rate. The second is to impose the volume fraction by fixing the distance between the plates. In this case, the pressure on the top plate typically increases with shear rate. These configurations give different results for the shear stress as a function of shear rate, but both are fully equivalent, as shown by da Cruz et al. (2005). We begin our discussion by considering the constant pressure case.

Friction and dilatancy laws. A crucial observation raised by da Cruz et al. (2005) and Lois et al. (2005) is that, in the simple sheared configuration for infinitely rigid particles, dimensional analysis strongly constrains the stress/shear rate relations. For large systems (i.e., when the distance between the plates plays no role), the system is controlled by a single dimensionless parameter called the inertial number:

$$I = \frac{\dot{\gamma}d}{\sqrt{P/\rho_p}}. \quad (1)$$

A possible interpretation of this parameter is given in terms of the ratio between two time scales (GDR MiDi 2004): (a) a microscopic time scale $d/\sqrt{P/\rho_p}$, which represents the time it takes for a particle to fall in a hole of size d under the pressure P and which gives the typical time scale of rearrangements, and (b) a macroscopic time scale $1/\dot{\gamma}$ linked to the mean deformation. Small values of I correspond to a quasi-static regime in the sense that macroscopic deformation is slow compared to microscopic rearrangement, whereas large values of I correspond to rapid flows. The dimensional analysis tells us that, to switch from a quasi-static to inertial regime, one can either increase the shear rate or decrease the pressure. This inertial number is also equivalent to the square root of the Savage number or Coulomb number introduced by some authors as the ratio of collisional stress to total stress (Ancey et al. 1999, Savage 1984). Importantly, this parameter is the only dimensionless number for rigid particles [this is not the case for soft particles, in which an elastic time scale comes into play (Campbell 2002, Shen & Sankaran 2004)]. As a consequence, for rigid grains, the shear stress is proportional to the pressure, with the effective friction coefficient and the volume fraction being functions of I :

$$\tau = P\mu(I) \quad \text{and} \quad \Phi = \Phi(I). \quad (2)$$

Figure 5a,b shows the functions $\mu(I)$ and $\Phi(I)$ measured in the discrete numerical simulations of da Cruz et al. (2005) carried out in two dimensions with discs (filled circles). The friction coefficient is nonzero for $I = 0$, increases with I , appears to saturate at higher inertial numbers, and eventually decreases when reaching the kinetic gas regime. The interparticle friction coefficient μ_p has little effect on the macroscopic friction, except when $\mu_p = 0$. The volume fraction $\Phi(I)$ decreases linearly over the range of inertial numbers investigated.

Pressure-controlled versus volume-controlled shearing. Carrying out shear experiments at controlled pressure is not common in rheology and needs some discussion. One can conduct the same kind of plane-shear computation at constant volume fraction Φ . In this case, the fact that no internal stress scale exists implies that

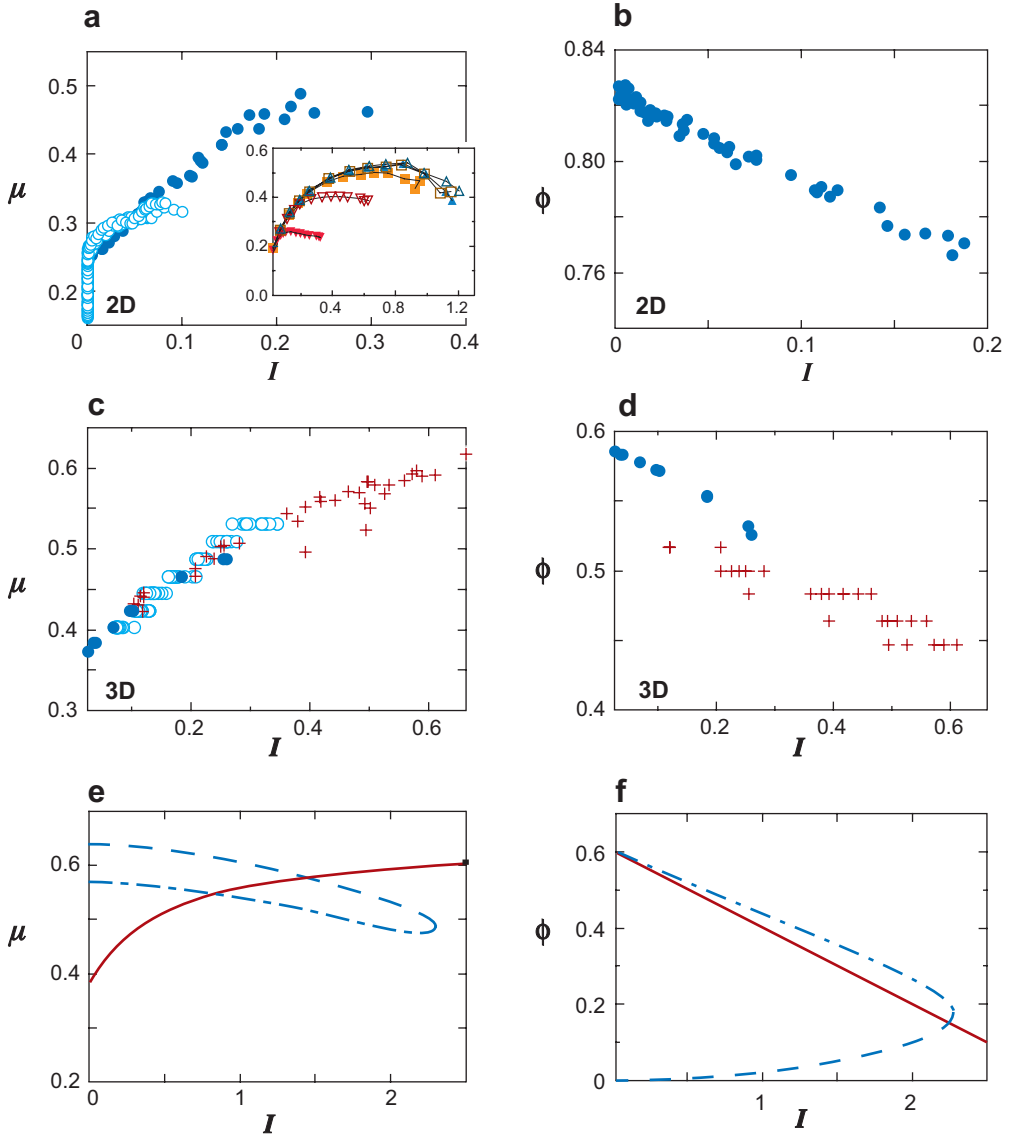
Figure 5

Friction coefficient and volume fraction as a function of the inertial number. (a, b) Data obtained on 2D flows. Filled circles represent plane-shear simulations at constant pressure (da Cruz et al. 2005); $e = 0.5$, and $\mu_p = 0.4$. Open circles represent rotating drum simulations in which the local friction coefficient and the local I parameter are obtained from velocity profiles at different rotation rates (Renouf et al. 2005). (*Inset in panel a*) Plane-shear simulation at constant volume fraction (G. Lois, A. Lemaitre & J.M. Carlson, submitted); $\mu_p = 0$, and $e = 0.9, 0.75, 0.5, 0.25, 0.1, 0$. (c, d) Data obtained on 3D flows. Open circles represent inclined-plane experiments, where $\mu(I)$ is derived from measurements of depth-averaged velocities at different inclinations and thicknesses (GDR MiDi 2004, Pouliquen 1999a); filled circles represent inclined-plane simulations (Baran et al. 2006); and crosses represent plane-shear experiments (material II, sample A), in which normal stress and volume fraction measurements were obtained in 3D annular shear cells (Savage & Sayed 1984). (e, f) Continuous line is fit given by Equations 4, and dashed line is the prediction of kinetic theory for frictionless spheres; $e = 0.6$ (Lun et al. 1984).

the shear stress and normal stress are given by what is called the Bagnold scaling:

$$\tau = \rho_p d^2 f_1(\Phi) \dot{\gamma}^2 \quad \text{and} \quad P = \rho_p d^2 f_2(\Phi) \dot{\gamma}^2. \quad (3)$$

This expression is not restricted to the collisional arguments initially given by Bagnold (1954), but simply comes from dimensional analysis and is valid for all shear rates (Lois et al. 2005). As a consequence, in a constant volume experiment, no threshold appears to exist, and τ goes to zero when $\dot{\gamma}$ goes to zero, although the ratio τ/P remains finite. It is important to notice that this description is identical to Expression 2 given for



pressure-controlled experiments. The following relations exist between $\mu(I)$ and $\Phi(I)$ and the functions f_1 and f_2 : $\Phi(I) = f_2^{-1}(1/I^2)$ and $\mu(I) = I^2 f_1(f_2^{-1}(1/I^2))$. The inset of **Figure 5a** shows data from the constant volume fraction simulations of G. Lois, A. Lemaitre & J.M. Carlson (submitted) plotted in terms of $\mu(I)$. The different curves correspond to different restitution coefficients e . A master curve $\mu(I)$ emerges, which corresponds to $e < 0.5$. For higher e , the friction law deviates from the master curve above a critical value of the inertial number I , which could be the signature of the transition to the kinetic regime.

Although the descriptions in Equations 2 and 3 are equivalent, we find the first more relevant for fluid mechanical treatments as emphasized by da Cruz et al. (2005). Indeed, the functions f_1 and f_2 diverge very rapidly close to the maximum volume fraction, which makes them difficult to measure accurately, whereas using the friction and dilatancy laws eliminates this difficulty. Moreover, in the dense flow regime, in which variations of the volume fraction are small, an incompressible assumption is possible within the framework of Equations 2: The dilatancy and friction laws are decoupled, which allows one to neglect the variations of Φ without losing the variations of the friction coefficient, which characterize the viscous nature of the material (Jop et al. 2006).

3.2. A Local Rheology

It is tempting to consider the friction and dilatancy laws obtained in plane shear as constitutive equations for dense granular flows. This appealing idea is not a priori justified, as nothing stipulates that stresses developing in inhomogeneous systems will be the same as in plane shear. This is true only for a local rheology, in which the shear stress depends only on the local shear rate and pressure.

Support for the local assumption is given in **Figure 5**, which presents collected data from other configurations and plots them in terms of friction and dilatancy laws. **Figure 5a,b** shows 2D measurements obtained both in plane-shear (da Cruz et al. 2005) and rotating-drum simulations (Renouf et al. 2005), whereas **Figure 5c,d** shows 3D data from simulations (Baran et al. 2006) and experiments (Pouliquen 1999a) of flows down inclined planes and from experiments in an annular shear cell (Savage & Sayed 1984). A striking collapse is observed, showing that the inertial number I remains the relevant parameter and that the friction law $\mu(I)$ is the same in different configurations. This result suggests that dense granular flows may indeed be described in terms of local friction and dilatancy laws.

As a result, Equations 2 are good candidates for constitutive laws. The functions $\mu(I)$ and $\Phi(I)$ can be fitted as follows (Jop et al. 2005, Pouliquen et al. 2006):

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1} \quad \text{and} \quad \Phi = \Phi_{max} + (\Phi_{min} - \Phi_{max})I. \quad (4)$$

Typical values of the constants obtained for monodispersed glass beads in three dimensions are $\mu_1 = \tan 21^\circ$, $\mu_2 = \tan 33^\circ$, $I_0 = 0.3$, $\Phi_{max} = 0.6$, and $\Phi_{min} = 0.4$. Those functional forms have not been tested for large values of the inertial number I . However, the choice of a friction law that saturates to a finite value μ_2 when I

goes to infinity is supported by experiments of steady granular fronts flowing down a slope (Pouliquen 1999b): At the tip of a front, the shear rate goes to infinity, whereas experiments reveal that the slope, and hence the friction coefficient, remains finite. This is consistent with the saturation of $\mu(I)$ to μ_2 .

3.3. Application to Different Configurations

In this section, we review different flow configurations and check the extent to which the experimental observations can be captured by the simple description in Equations 2.

Inclined plane. Researchers have investigated flows of grains on rough surfaces (Figure 4d) both in experiments and in numerical simulations using discrete element methods (GDR MiDi 2004, and references therein). The principal observations for steady uniform flows down rough inclined planes are the following: (a) The volume fraction is constant across the layer; (b) for thick layers, the velocity follows the so-called Bagnold profile and varies with the depth to the power $3/2$; (c) for thin layers, when b is close to the minimum thickness for flow, the velocity profile is close to linear (Rajchenbach 2003, Silbert et al. 2003); and (d) there is evidence of an empirical flow rule, where the depth-averaged velocity $\langle u \rangle$ is related to the thickness of the layer and to the inclination by an empirical relation $\langle u \rangle / \sqrt{gb} \propto b / h_{stop}(\theta)$, where $h_{stop}(\theta)$ is the minimum thickness for flow (Deboeuf et al. 2006, Pouliquen 1999a, Silbert et al. 2003).

Many of these observations are captured by the local rheology. For inclined planes, a force balance implies that the friction coefficient is constant across the layer and equal to $\tan \theta$. Equations 2 thus predict that the parameter I and the volume fraction Φ are constant across the layer. Moreover, the definition of I (Equation 1) implies for a constant I that the shear rate varies like the square root of the pressure, implying a Bagnold velocity profile and a mean velocity varying as the thickness to the power $3/2$.

However, a few observations are not predicted by the local rheology. First, the friction law does not recover linear profiles for thin layers. Second, close to the free surface or the bottom, simulations show that I is not constant, contradicting the prediction (Baran et al. 2006). Finally, the transition between solid and liquid behavior is not correctly captured. In Section 2.1 we discuss that the minimum angle θ_{stop} above which a flow is possible depends on the thickness b . The local rheology predicts a simpler behavior, with a critical angle independent of the thickness and equal to $\tan^{-1} \mu_1$. However, assuming the local rheology (Equations 2), one can easily show that the function $\mu(I)$ is related to the function $\theta_{stop}(b)$ (GDR MiDi 2004). Whether this correlation between the shape of the flow threshold and the friction law is just a coincidence or results from more fundamental physical reasons remains an open question.

Flows on a pile. Flows on a pile are obtained when grains are poured on top of a static packing (Figure 4e). In this configuration, steady uniform flows can be obtained for long enough piles. Contrary to the inclined plane, the slope of the pile is not chosen by

the experimentalist, but self-adjusts (Taberlet et al. 2003). The only control parameter is then the injection flow rate. In this configuration, a localized free surface flow develops with a velocity profile approximately linear in the upper region, followed by an exponential creeping tail below (Komatsu et al. 2001). Jop et al. (2005) have shown recently that the localization of the flow on the free surface in this geometry is entirely controlled by the lateral wall, which introduces additional friction. The thickness of the flowing layer then scales with the distance between the lateral walls (Jop et al. 2005).

The local rheology leads to quantitative predictions for this configuration as well. First, the scaling with I in the rheology implies a nontrivial scaling for how the inclination, free surface velocity, and flow thickness depend on the flow rate and channel width. Second, quantitative predictions are made without any fitting parameter once the friction law is determined independently in inclined-plane experiments (Jop et al. 2005).

However, some observations cannot be explained within the local rheology. First, the creeping exponential tail observed at the boundary between the liquid layer and the solid pile is not captured. Second, the flow threshold is not correctly described. Experimentally, a critical flow rate exists below which continuous flow is not observed: The flow is intermittent, consisting of successive transient avalanches (Jop et al. 2005, Lemieux & Durian 2000). The local rheology predicts a continuous steady flow even for very low flow rates. The last disagreement concerns high flow rates, at which a dilute region develops on top of the dense flow and high inclinations up to 60° have been observed (Louge et al. 2005, Taberlet et al. 2003). In contrast, the model predicts that the slope in a steady uniform regime is always below $\tan^{-1} \mu_2$ (Jop et al. 2005).

Rotating drums. Rotating drums (Figure 4f) have been the subject of many experimental studies and a few numerical ones (GDR MiDi 2004, and references therein). This configuration has similarities with flow on piles, as a localized free surface flow is observed on top of a region experiencing a rigid rotation with the drum. However, the data are more difficult to interpret, as the flow is nonuniform (Renouf et al. 2005). The flowing layer exhibits a lens shape, resulting from the exchange of grains along the solid-liquid interface. Moreover, experiments are often carried out in thin drums, for which sidewall friction cannot be neglected. Despite these intrinsic difficulties, two recent attempts have been made to check the applicability of local rheology. Renouf et al. (2005) have shown in 2D simulations that the friction law $\mu(I)$ is locally satisfied along the profile (Figure 5a). Orpe & Khakhar (2007) have shown in experiments that the variation of the coefficient of friction is also embedded in the I parameter. However, owing to the nonuniformity of the flow, a tensorial generalization of Equation 2 should be used to test the local rheology in this configuration. We discuss this point further in Section 3.4.

Confined flows. Researchers have studied other confined geometries in addition to plane shear, including the cylindrical Couette cell (Figure 4b), the vertical silo (Figure 4c), and plane shear with gravity (GDR MiDi 2004, and references therein). In all these geometries, the velocity profile is localized in a shear band five to ten

particles thick located close to the moving wall. In more complex 3D geometries such as the modified Couette cell, in which the bottom is split into a rotating and a static part, shear zones with up to 40-particle diameters are observed (Fenistein et al. 2004).

When applying the local rheology to these systems, one predicts the formation of shear bands, resulting from the nonuniformity of the stress distribution (Pouliquen et al. 2006). However, the thickness of the predicted shear bands depends on the shear velocity and vanishes in the quasi-static limit. This contradicts observations and clearly shows that the local rheology cannot capture the quasi-static regime.

3.4. Tensorial Formulation

The relative success of the local rheology in quantitatively predicting some simple sheared configurations motivates us to analyze more complex configurations such as the dam-break problem, spreading of masses, and free surface instabilities. In these cases, the flows are characterized by shears in different directions. The comparison with the local rheology thus requires the generalization of the simple scalar laws (Equations 2) to a tensorial formulation. Following the work of several previous authors (Depken et al. 2006, Goddard 1986, Savage 1983, Schaeffer 1987), one can assume a colinearity between the shear stress and the shear rate tensors. Moreover, with the assumption that the volume fraction is constant in the limit of dense systems, the constitutive law of the granular liquid takes the form of a visco-plastic law, in which the stress tensor is given by (Jop et al. 2006)

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}, \quad (5)$$

where P is the isotropic pressure,

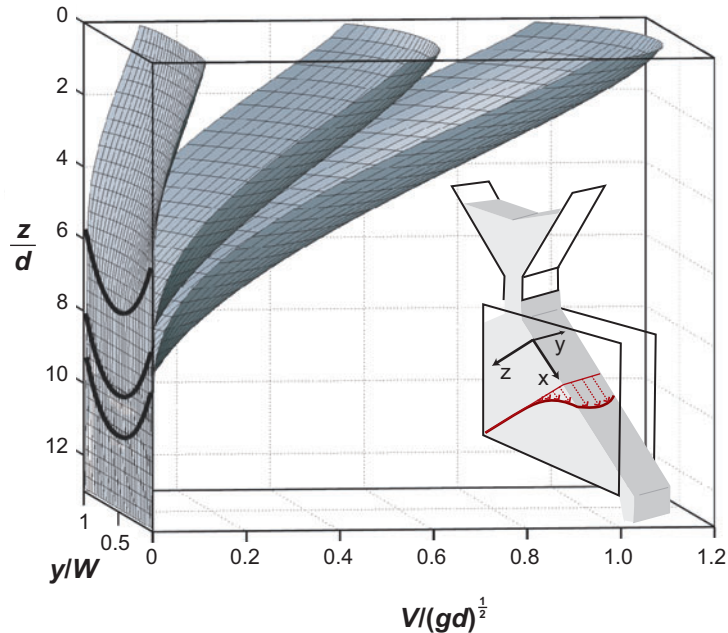
$$\tau_{ij} = \eta\dot{\gamma}_{ij}, \quad \text{with} \quad \eta = \frac{\mu(I)P}{|\dot{\gamma}|}, \quad (6)$$

where $|\dot{\gamma}|$ is second invariant of the shear rate tensor: $|\dot{\gamma}| = \sqrt{\frac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}$.

The granular liquid is then described as a peculiar non-Newtonian incompressible fluid, with a viscosity η depending both on the shear rate and the pressure, a signature of the underlying frictional nature of the medium. When $|\dot{\gamma}|$ goes to zero, the viscosity diverges, and one recovers a flow threshold given by a frictional Drucker-Prager criterion: $|\tau| > \mu_1 P$. In this constitutive law, no normal stress difference exists, consistent with simulations for plane shear (da Cruz et al. 2005) and flow down inclined planes (Silbert et al. 2001). We have tested this tensorial rheology in two configurations: (a) free surface flows between rough walls (Jop et al. 2006), in which shear in two directions is present, and (b) the long wave instability observed for flows down an inclined plane (Forterre 2006). In both cases, striking quantitative agreement was obtained between predictions and experimental measurements for velocity profiles and dispersion relations. To further test the relevance and limits of the local rheology, it would be interesting to study other configurations such as avalanche triggering (**Figure 6**), the dam-break problem, or the rotating drum. However, simulating such a visco-plastic law with its divergence of viscosity has proven to be a rather tricky exercise (Frigaard & Nouar 2005).

Figure 6

Temporal evolution of the 3D velocity profile predicted by Equations 5 and 6 for the start of granular flow in a narrow channel.



3.5. Limits of the Local Approach: Beyond Dimensional Analysis

The simple local rheology seems to provide a minimal framework to describe many features of dense granular flows. However, the link with microscopic grains properties is still lacking, and serious limitations exist. In this section we briefly discuss these difficulties and present recent theoretical approaches that have been proposed to overcome them.

Microscopic origin. So far, the presented approach is phenomenological, and apart from dimensional analysis, the shape of the friction law is measured and not derived from the microscopic properties of the grains. Investigators have made several attempts to connect the rheology to the evolution of the distribution of contacts within the granular assembly (Ancey et al. 1999, da Cruz et al. 2005, Lois et al. 2006). It seems that the origin of the increase of friction with the inertial number is linked to a modification of the anisotropy in the contact distribution. Interestingly, a similar shape of the friction law is obtained when investigating the motion of a single grain on a bumpy surface (B. Andreotti, submitted).

Quasi-static limits. The second important limitation of this visco-plastic approach concerns quasi-static flows: Shear bands observed in confined systems are not correctly described. To capture the correct behavior in this shear rate-independent regime, a first approach consists of modifying plasticity models. Mohan et al. (2002) have developed a modified Cosserat approach, in which an additional degree of

freedom is introduced through local rotation. Kamrin & Bazant (2007) have recently proposed another model inspired by Mohr-Coulomb plasticity. It consists of introducing a stochastic flow rule in a Coulomb material. The last class of models is inspired by the plasticity of amorphous solids (Lemaitre 2002) and is based on the definition of elementary plastic processes.

A second approach to describing quasi-static flows consists of explicitly writing nonlocal equations. This is motivated by observations of large spatial correlations close to the flow threshold, as seen in force networks (Lois et al. 2006, Radjai & Roux 2003) and velocity fluctuations (Baran et al. 2006, Bonamy et al. 2002, Pouliquen 2004). Trying to incorporate such nonlocal effects in constitutive laws is not an easy task and is a problem encountered in many other physical systems exhibiting a jamming transition, but several attempts have been made. Mills et al. (1999) have proposed constitutive equations in which stresses explicitly result from integrals over force chains. Pouliquen et al. (2001) have proposed a self-activated process, in which shear deformations at a point induce fluctuations that may trigger shears at some other position. All these models have achieved moderate success, but they do not give a unified description of the transition between the quasi-static and liquid regimes.

Flow threshold. The third important limitation of the visco-plastic approach concerns the flow threshold. In the model, the flow threshold is given by a simple Coulomb criterion, whereas hysteresis and finite size effects exist, as discussed in Section 2.1. Capturing the hysteretic character of granular flows is the major motivation of a model developed by Aranson & Tsimring (2002). The granular media is described as the mixture of solid and liquid phases, whose relative fraction is controlled by a Landau equation. Some attempts have been made to connect this approach to microscopic measurements (Volfson et al. 2003). The model describes nontrivial behaviors observed when triggering avalanches on an inclined plane but fails to predict the correct rheology (Aranson & Tsimring 2006).

Transition to the kinetic regime. The last limitation, which has been much less explored, concerns the transition to the kinetic gaseous regime. Configurations such as flow on a pile at very high flow rate clearly show a gaseous layer forming on top of a liquid layer (Louge et al. 2005). This gaseous regime is not captured in the simple visco-plastic approach. Conversely, the original kinetic theory based on binary collisions does not capture the correct behavior in the dense regime as shown in **Figure 5e,f**. The predicted $\mu(I)$ curve exhibits two branches, the bottom one corresponding to dense flows. The friction coefficient decreases with I , in contradiction with observation.

This discrepancy has motivated several attempts to modify the original kinetic theory to account for enduring contacts. The first consists of writing the shear stress as the sum of a frictional term and a collisional term (Johnson & Jackson 1987, Josserand et al. 2006, Louge 2003, Savage 1983). However, in our opinion, the introduction of a rate-independent pressure term in the equation of state poses problems from a dimensional point of view. A second class of approaches consists of modifying the

transport coefficients of the kinetic theory. Bocquet et al. (2001) propose to change the expression for the viscosity by analogy with hard-sphere glasses. Savage (1998) proposes to modify both the viscosity and dissipation terms. Kumaran (2006) seeks higher-order Burnett terms in the Boltzmann equation development to recover the correct variation of the volume fraction with inclination for flows down inclined planes. A last idea postulates the existence of a length scale larger than the particle diameter related to the formation of clusters. Ertas & Halsey (2002) and Wang (2004) have developed a simple heuristic mixing length theory. Recently, Jenkins (2006) introduced a correlation length in the dissipation term of the classical kinetic theory and can predict important features observed for flows down inclined planes.

Considering the large spectrum of theoretical approaches, we can infer that no consensus exists and that finding constitutive laws valid from the quasi-static to dilute regimes remains a serious challenge. In configurations in which the flowing layer is thin, another theoretical framework based on depth-averaged equations has been proposed. The question of the constitutive laws is then made much simpler as it reduces to an interfacial law between the granular layer and the bottom.

4. SHALLOW-WATER DESCRIPTION

Savage & Hutter (1989) introduced depth-averaged or Saint-Venant equations in the context of granular flows. The initial motivation was to model natural hazards such as landslides or debris flows (Heinrich et al. 2001, Naaïm et al. 1997). Assuming that the flow is incompressible and the spatial variation of the flow takes place on a scale larger than the flow thickness, one obtains the Saint-Venant equations by integrating the 3D mass and momentum conservation equations. For 2D flow down a slope making an angle θ with the horizontal (see **Figure 7**), the depth-averaged equations

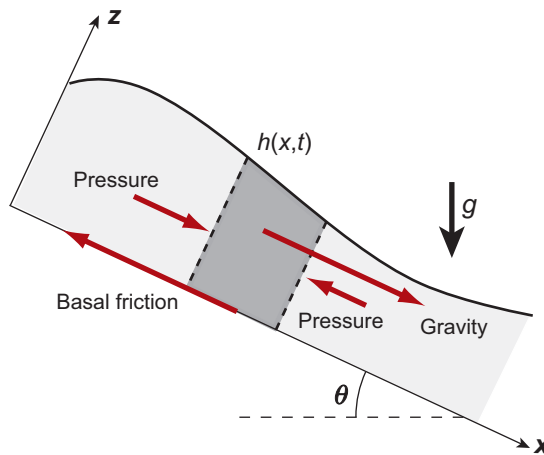


Figure 7

Forces balance in the shallow-water description.

reduce to

$$\frac{\partial b}{\partial t} + \frac{\partial b \langle u \rangle}{\partial x} = 0, \quad (7)$$

$$\rho_p \phi \left(\frac{\partial b \langle u \rangle}{\partial t} + \alpha \frac{\partial b \langle u \rangle^2}{\partial x} \right) = \left(\tan \theta - \mu_b - K \frac{\partial b}{\partial x} \right) \rho_p \phi g b \cos \theta, \quad (8)$$

where b is the local flow thickness, $\langle u \rangle = Q/b$ is the depth-averaged velocity (Q being the flow rate per unit of width), and ϕ is the volume fraction, assumed constant.

Equation 7 is the mass conservation, and Equation 8 is the momentum equation in which the acceleration is balanced by three forces (**Figure 7**): the gravity parallel to the plane, the tangential stress between the fixed bottom and the flowing layer (written as a basal friction coefficient μ_b multiplied by the normal stress), and a pressure force related to the thickness gradient. The coefficient α is related to the assumed velocity profile across the layer and is of order 1. The coefficient K represents the ratio of the normal horizontal stress (x direction) to the normal vertical stress (z direction) and is close to 1 for steady uniform flows (Silbert et al. 2001). The main advantage of the Saint-Venant equations is that the dynamics of the flowing layer can be predicted without knowing in detail the internal structure of the flow. The complex 3D rheology of the material is mainly embedded in the basal friction term μ_b . Taking a simple constant Coulomb-like basal friction is sometimes sufficient to capture the main flow characteristics and has been used to describe granular slumping (Balmforth & Kerswell 2005, Lajeunesse et al. 2005, Mangeney-Castelnaud et al. 2005), rapid flows down smooth inclines (Greve et al. 1994, Wieland et al. 1999), and shock waves (Gray et al. 2003, Hokanardottir & Hogg 2005). However, for flows down rough inclines, the assumption of a constant solid friction is not compatible with the observation of steady uniform flows over a range of inclination angles. We have proposed more complex basal friction laws $\mu_b(\langle u \rangle, b)$ that lead to quantitative predictions in complex situations such as a propagating steady front, mass spreading, or surface instabilities (Forterre & Pouliquen 2003, Mangeney-Castelnaud et al. 2007, Pouliquen 1999a, Pouliquen & Forterre 2002). It should be noted that the Saint-Venant equations (Equations 7 and 8) represent a first-order development in terms of the flow aspect ratio. Therefore, they do not capture second-order effects such as longitudinal and lateral momentum diffusion, which stabilize instabilities (Forterre 2006) and control lateral stresses. The knowledge of the full 3D constitutive equations (Equations 4 and 5) may allow the development of more complex depth-averaged models (Balmforth & Liu 2004, Ruyer-Quil & Manneville 2000).

Another application of the depth-averaged equations concerns situations in which the flowing layer propagates on an erodible layer, such as flow on top of a static pile. In this case, exchange of matter exists between the liquid and solid phase. An additional equation is then needed to determine the solid-liquid interface. Several closures have been proposed (Aradian et al. 2002). The first model (Bouchaud et al. 1994, Boutreux et al. 1998) assumes that erosion/deposition is controlled by the difference between the local slope and the critical pile angle. Other approaches assume a relation between the average velocity and the flow thickness, either by fixing the velocity gradient (Douady et al. 1999) or by prescribing a basal shear stress at the solid/liquid boundary

(Khakhar et al. 2001). These models predict qualitatively nontrivial behaviors such as avalanche front propagation (Douady et al. 2001, Taberlet et al. 2004). Although these two-layer approaches seem to be a promising framework to study avalanching flows on erodible beds, the closures proposed to date are not compatible with observations of steady uniform flow on pile, in which the flowing thickness is selected by the sidewalls (Jop et al. 2005). This clearly shows that a proper development of shallow-water models has to rely on the knowledge of the full constitutive equations, a goal not completely achieved.

5. CONCLUSION

In this review, we present a brief survey of our current understanding of dense granular flows. Our main intention was to emphasize that a zero-order description of the viscous-like behavior of dense granular flows is now available, which relies on simple but solid dimensional arguments. A frictional visco-plastic formulation has been developed that gives quantitative predictions for different flow configurations and can serve as an initial tool to predict other configurations encountered in applications. Although promising, this approach fails to capture the details of the quasi-static flows and the transition to solid or gaseous regimes. It is difficult to anticipate that more elaborate constitutive equations will be developed in the near future that can describe the whole phenomenology of granular flows. The diversity of the theoretical approaches clearly shows that the task is difficult, the central question now being, in our opinion, how to account for the nonlocal effects created by the network of enduring contacts.

The description of more complex materials than the simple dry monodispersed spheres discussed above represents important challenges. Irregular-shaped particles, cohesion, interaction with interstitial fluids, and polydispersity are often encountered in most of the real granular materials used in industries or occurring in geophysics. Few studies tackle these questions in light of the recent rheology found for simple dry cohesionless materials. The role of the grain shape has been considered in few studies (Börzsönyi et al. 2005, GDR MiDi 2004). It seems that nonlocal effects are enhanced owing to the irregular shape, leading to a narrower range of validity of the local rheology. Investigators have also recently studied the role of cohesion based on dimensional arguments and have proposed a friction law that depends on both the inertial number and a second dimensionless number measuring the relative strength of cohesion to confining pressure (Rognon et al. 2006). The role of the interstitial fluid has been also investigated in submarine granular avalanches. Interestingly, the same friction law as that for the dry case seems to hold when the typical time scale of microscopic rearrangements is changed to a viscous time scale (Cassar et al. 2005). This result opens new perspectives at the boundary between granular media and dense suspensions (Stickel & Powell 2005). Finally, the problem of polydispersity and segregation has been intensively studied (Ottino & Khakhar 2000). Researchers have identified important mechanisms (Felix & Thomas 2005, Savage & Lun 1988) and developed models to address specific configurations. There is no doubt that ongoing progress in the rheology of granular flows will help in describing polydispersed systems.

DISCLOSURE STATEMENT

The authors are not aware of any biases that might be perceived as affecting the objectivity of this review.

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LITERATURE CITED

- Ancey C, Coussot P, Evesque P. 1999. A theoretical framework for granular suspension in a steady simple shear flow. *J. Rheol.* 43:1673–99
- Aradian A, Raphaël E, de Gennes PG. 2002. Surface flows of granular materials: a short introduction to some recent models. *C. R. Phys.* 3:187–96
- Aranson IS, Tsimring LS. 2002. Continuum theory of partially fluidized granular flows. *Phys. Rev. Lett.* 65:061303
- Aranson IS, Tsimring LS. 2006. Patterns and collective behavior in granular media: theoretical concepts. *Rev. Mod. Phys.* 78:641–92
- Bagnold RA. 1954. Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear. *Proc. R. Soc. London Ser. A* 225:49–63
- Balmforth NJ, Kerswell RR. 2005. Granular collapse in two dimensions. *J. Fluid Mech.* 538:399–428
- Balmforth NJ, Liu JJ. 2004. Roll waves in mud. *J. Fluid Mech.* 519:33–54
- Baran O, Ertas D, Halsey TC, Grest GS, Lechman JB. 2006. Velocity correlations in dense gravity-driven granular chute flow. *Phys. Rev. E* 74:051302
- Bocquet L, Losert W, Schalk D, Lubensky TC, Gollub JP. 2001. Granular shear flow dynamics and forces: experiments and continuum theory. *Phys. Rev. E* 65:011307
- Bonamy D, Daviaud F, Laurent L, Bonetti M, Bouchaud JP. 2002. Multi-scale clustering in granular surface flows. *Phys. Rev. Lett.* 89:034301
- Börzsönyi T, Halsey T, Ecke RE. 2005. Two scenarios for avalanche dynamics in inclined granular layers. *Phys. Rev. Lett.* 94:208001
- Bouchaud JP, Cates M, Prakash JR, Edwards SF. 1994. A model for the dynamics of sandpile surfaces. *J. Phys.* 14:1383–410
- Boutreux T, Raphaël E, de Gennes PG. 1998. Surface flows of granular material: a modified picture for thick avalanches. *Phys. Rev. E* 58:4692–700
- Campbell CS. 1990. Rapid granular flows. *Annu. Rev. Fluid Mech.* 22:57–92
- Campbell CS. 2002. Granular shear flows at the elastic limit. *J. Fluid Mech.* 465:261–91
- Cassar C, Nicolas M, Pouliquen O. 2005. Submarine granular flows down inclined planes. *Phys. Fluids* 17:103301
- Courrech du Pont S, Gondret P, Perrin B, Rabaud M. 2003. Wall effects on granular heap stability. *Eur. Lett.* 61:492–98

- da Cruz F, Chevoir F, Bonn D, Coussot P. 2002. Viscosity bifurcation in granular materials, foams, and emulsions. *Phys. Rev. E* 66:051305
- da Cruz F, Chevoir F, Roux JN, Iordanoff I. 2004. Macroscopic friction of dry granular materials. *Tribol. Ser.* 43:53–61
- da Cruz F, Emam S, Prochnow M, Roux JN, Chevoir F. 2005. Rheophysics of dense granular materials: discrete simulation of plane shear flows. *Phys. Rev. E* 72:021309
- Daerr A, Douady S. 1999a. Sensitivity of granular surface flows to preparation. *Europhys. Lett.* 47:324–30
- Daerr A, Douady S. 1999b. Two types of avalanche behavior in granular media. *Nature* 399:241–43
- Deboeuf S, Lajeunesse E, Dauchot O, Andreotti B. 2006. Flow rule, self-channelization and levees in unconfined granular flows. *Phys. Rev. Lett.* 97:158303
- Depken M, van Saaloos W, van Hecke M. 2006. Continuum approach to wide shear zones in quasistatic granular matter. *Phys. Rev. E* 73:031302
- Douady S, Andreotti B, Cladé P, Daerr A. 2001. The four avalanche fronts: a test case for granular surface flow modeling. *Adv. Complex Syst.* 4:509–22
- Douady S, Andreotti B, Daerr A. 1999. On granular surface flow equations. *Eur. Phys. J. B* 11:131–42
- Ertas D, Halsey TC. 2002. Granular gravitational collapse and chute flow. *Eurphys. Lett.* 60:931–37
- Felix G, Thomas N. 2004. Relation between dry granular flow regimes and morphology of deposits: formation of levees in pyroclastic deposits. *Earth Planet. Sci. Lett.* 221:197–213
- Felix G, Thomas N. 2005. Evidence of two effects in the size segregation process in dry granular media. *Phys. Rev. E* 70:051307
- Fenistein D, van de Meent JW, van Hecke M. 2004. Universal and wide shear zones in granular bulk flow. *Phys. Rev. Lett.* 92:094301
- Forterre Y. 2006. Kapiza waves as a test for three-dimensional granular flow rheology. *J. Fluid Mech.* 563:123–32
- Forterre Y, Pouliquen O. 2002. Stability analysis of rapid granular chute flows: formation of longitudinal vortices. *J. Fluid Mech.* 467:361–87
- Forterre Y, Pouliquen O. 2003. Long-surface-wave instability in dense granular flows. *J. Fluid Mech.* 486:21–50
- Frigaard IA, Nour C. 2005. On the usage of viscosity regularisation methods for visco-plastic fluid flow computation. *J. Non-Newton. Fluid Mech.* 127:1–26
- GDR MiDi. 2004. On dense granular flows. *Eur. Phys. J. E* 14:341–65
- Goddard JD. 1986. Dissipative materials as constitutive models for granular media. *Acta Mech.* 83:3–13
- Goldhirsch I. 2003. Rapid granular flows. *Annu. Rev. Fluid Mech.* 35:267–93
- Gray JMNT, Tai YC, Noelle S. 2003. Shock waves, dead zones and particle-free regions in rapid granular free-surface flows. *J. Fluid Mech.* 491:161–81
- Greve R, Koch T, Hutter K. 1994. Unconfined flow of granular avalanches along a partly curved surface: experiments and theoretical predictions. *Proc. R. Soc. London Ser. A* 342:573–600

- Heinrich P, Piatanesi A, Hebert. 2001. Numerical modelling of tsunami generation and propagation from submarine slumps: the 1998 Papua New Guinea event. *Geophys. J. Int.* 145:97–111
- Hill G, Yeung S, Koehler SA. 2005. Scaling vertical drag forces in granular media. *Europhys. Lett.* 72:137–43
- Hokanardotir KM, Hogg AJ. 2005. Oblique shocks in rapid granular flows. *Phys. Fluids* 17:077101
- Jaeger HM, Nagel SR, Behringer RP. 1996. Granular solids, liquids, and gases. *Rev. Mod. Phys.* 68:1259–73
- Jenkins JT. 2006. Dense shearing flows of inelastic disks. *Phys. Fluids* 18:103307
- Jenkins JT, Richman MW. 1985. Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks. *Phys. Fluids* 12:3485–94
- Jenkins JT, Savage SB. 1983. A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles. *J. Fluid Mech.* 130:187–202
- Johnson PC, Jackson R. 1987. Frictional-collisional constitutive relations for granular materials, with application to plane shearing. *J. Fluid Mech.* 176:67–93
- Jop P, Forterre Y, Pouliquen O. 2005. Crucial role of sidewalls in granular surface flows: consequences for the rheology. *J. Fluid Mech.* 541:167–92
- Jop P, Forterre Y, Pouliquen O. 2006. A constitutive law for dense granular flows. *Nature* 441:727–30
- Josserand C, Lagre PY, Lhuillier D. 2006. Granular pressure and the thickness of a layer jamming on a rough incline. *Europhys. Lett.* 73:363–69
- Kamrin K, Bazant MZ. 2007. A stochastic flow rule for granular materials. *Phys. Rev. E*. In press
- Khakhar DV, Orpe AV, Andersen P, Ottino JM. 2001. Surface flow of granular materials: model and experiments in heap formation. *J. Fluid Mech.* 441:255–64
- Komatsu TS, Inagaki S, Nakagawa N, Nasuno S. 2001. Creep motion in a granular pile exhibiting steady surface flow. *Phys. Rev. Lett.* 86:1757–60
- Kumaran V. 2006. The constitutive relation for the granular flow of rough particles, and its application to the flow down an inclined plane. *J. Fluid Mech.* 561:1–42
- Lajeunesse E, Mangeney-Castelnau A, Vilotte JP. 2004. Spreading of a granular mass on a horizontal plane. *Phys. Fluids* 16:2371–81
- Lajeunesse E, Monnier JB, Homsy GM. 2005. Granular slumping on a horizontal surface. *Phys. Fluids* 17:103302
- Lemaitre A. 2002. Origin of a repose angle: kinetics of rearrangement for granular materials. *Phys. Rev. Lett.* 89:064303
- Lemieux PA, Durian DJ. 2000. From avalanches to fluid flow: a continuum picture of grain dynamics down a heap. *Phys. Rev. Lett.* 85:4273–76
- Liu AJ, Nagel SR. 1998. Jamming is not just cool any more. *Nature* 396:21–22
- Lois G, Lemaitre A, Carlson J. 2005. Numerical tests of constitutive laws for dense granular flows. *Phys. Rev. E* 72:051303
- Lois G, Lemaitre A, Carlson JM. 2006. Emergence of multi-contact interactions in contact dynamics simulations of granular shear flows. *Europhys. Lett.* 76:318–24
- Losert W, Bocquet L, Lubensky TC, Gollub JP. 2000. Particle dynamics in sheared granular matter. *Phys. Rev. Lett.* 85:1428–31

- Louge MY. 2003. Model for dense granular flows down bumpy inclines. *Phys. Rev. E* 67:061303
- Louge M, Valance A, Taberlet N, Richard P, Delannay R. 2005. Volume fraction profile in channeled granular flows down an erodible incline. *Proc. Powders Grains, Stuttgart, Ger.* Leiden: Balkema Publ.
- Lube G, Huppert HE, Sparks RSJ, Hallworth M. 2004. Axisymmetric collapse of granular columns. *J. Fluid Mech.* 508:175–99
- Lun CKK, Savage SB, Jeffrey DJ, Chepuruiy N. 1984. Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flow field. *J. Fluid Mech.* 140:223–56
- Mangeny-Castelnaud A, Bouchut F, Thomas N, Vilotte JP, Bristeau M-O. 2007. Numerical modeling of self-channeling granular flows and of their levee/channel deposits. *J. Geophys. Res.* 112:F02017
- Mangeny-Castelnaud A, Bouchut F, Vilotte JP, Lajeunesse E, Aubertin A, Pirulli M. 2005. On the use of Saint-Venant equations to simulate the spreading of a granular mass. *J. Geophys. Res.* 110:B09103
- Mills P, Loggia D, Tixier M. 1999. Model for a stationary dense granular flow along an inclined wall. *Europhys. Lett.* 45:733–38
- Mitarai N, Nakanishi H. 2004. Linear stability analysis of rapid granular flow on a slope and density wave formation. *J. Fluid Mech.* 507:309–34
- Mohan LS, Rao KK, Nott PR. 2002. A frictional Cosserat model for the slow shearing of granular materials. *J. Fluid Mech.* 457:377–409
- Mueth DM, Debregeas GF, Karczmar GS, Eng PJ, Nagel SR, Jaeger HM. 2000. Signatures of granular microstructure in dense shear flows. *Nature* 406:385–89
- Naaïm M, Vial S, Couture R. 1997. Saint-Venant approach for rock avalanches modelling. *Multiple Scale Analyses Coupled Phys. Syst.: Saint-Venant Symp.* Paris: Presse École Natl. Ponts Chaussées
- Orpe A, Khakhar DV. 2007. Rheology of surface granular flows. *J. Fluid Mech.* 571:1–32
- Ottino JM, Khakhar DV. 2000. Mixing and segregation of granular materials. *Annu. Rev. Fluid Mech.* 32:55–91
- Pouliquen O. 1999a. Scaling laws in granular flows down rough inclined planes. *Phys. Fluids* 11:542–48
- Pouliquen O. 1999b. On the shape of granular fronts down rough inclined planes. *Phys. Fluids* 11:1956–58
- Pouliquen O. 2004. Velocity correlations in dense granular flows. *Phys. Rev. Lett.* 93:248001
- Pouliquen O, Cassar C, Jop P, Forterre Y, Nicolas M. 2006. Flow of dense granular material: towards simple constitutive laws. *J. Stat. Mech.* 2006:P07020
- Pouliquen O, Chevoir F. 2002. Dense flows of dry granular material. *C. R. Phys.* 3:163–75
- Pouliquen O, Forterre Y. 2002. Friction law for dense granular flows: application to the motion of a mass down a rough inclined plane. *J. Fluid Mech.* 453:133–51
- Pouliquen O, Forterre Y, Ledizes S. 2001. Slow dense granular flows as a self-induced process. *Adv. Complex Syst.* 4:441–50

- Pouliquen O, Renaut N. 1996. Onset of granular flows on an inclined rough surface: dilatancy effects. *J. Phys. II* 6:923–35
- Quartier L, Andreotti B, Douady S, Daerr A. 2000. Dynamics of a grains on a sandpile model. *Phys. Rev. E* 62:8299–307
- Radjai F, Roux S. 2003. Turbulent-like behavior in quasi-static flow of granular media. *Phys. Rev. Lett.* 89:064302
- Rajchenbach J. 2003. Dense, rapid flows of inelastic grains under gravity. *Phys. Rev. Lett.* 90:144302
- Renouf M, Bonamy D, Dubois F, Alart P. 2005. Numerical simulation of two-dimensional steady granular flows in rotating drum: on surface flow rheology. *Phys. Fluids* 17:103303
- Rognon PG, Roux J-N, Wolf D, Naaim M, Chevoir F. 2006. Rheophysics of cohesive granular materials. *Europhys. Lett.* 74:644–50
- Roux J-N, Combes G. 2002. Quasistatic rheology and the origin of strain. *C. R. Phys.* 3:131–40
- Roux S, Radjai F. 1998. Texture-dependent rigid-plastic behavior. In *Physics of Dry Granular Media*, ed. HJ Hermann, JP Hovi, S Luding, pp. 229–36. Dordrecht: Kluwer Acad.
- Ruyer-Quil C, Manneville P. 2000. Improved modeling of flows down inclined planes. *Eur. Phys. J. B* 15:357–69
- Savage SB. 1983. Granular flows down rough inclines: review and extension. In *Mechanics of Granular Materials: New Models and Constitutive Equations*, ed. JT Jenkins, M Satake, pp. 261–82. Amsterdam: Elsevier
- Savage SB. 1984. The mechanics of rapid granular flows. *Adv. Appl. Mech.* 24:28966
- Savage SB. 1998. Analyses of slow high-concentration flows of granular materials. *J. Fluid Mech.* 377:1–26
- Savage SB, Hutter K. 1989. The motion of a finite mass of granular material down a rough incline. *J. Fluid Mech.* 199:177–215
- Savage SB, Lun CKK. 1988. Particle size segregation in inclined chute flow of dry cohesionless granular solids. *J. Fluid Mech.* 189:311–35
- Savage SB, Sayed M. 1984. Stresses developed by dry cohesionless granular materials sheared in an annular shear cell. *J. Fluid Mech.* 142:391–430
- Schaeffer DG. 1987. Instability in the evolution equations describing incompressible granular flow. *J. Differ. Equat.* 66:19–50
- Schofield A, Wroth P. 1968. *Critical State Soil Mechanics*. London: McGraw-Hill
- Shen HH, Sankaran B. 2004. Internal time scales in a simple shear granular flow. *Phys. Rev. E* 70:051308
- Silbert LE, Ertas D, Grest GS, Halsey TC, Levine D, Plimpton SJ. 2001. Granular flow down an inclined plane: Bagnold scaling and rheology. *Phys. Rev. E* 64:051302
- Silbert LE, Landry JW, Grest GS. 2003. Granular flow down a rough inclined plane: transition between thin and thick piles. *Phys. Fluids* 15:1–10
- Stickel JJ, Powell RL. 2005. Fluid mechanics and rheology of dense suspensions. *Annu. Rev. Fluid Mech.* 37:129–49

- Swinney HL, Rericha EC. 2004. Pattern formation and shock waves in granular gases. In *The Physics of Complex Systems (New Advances and Perspectives): Proc. Int. School Phys. Enrico Fermi, Course CLV 155, 173*, ed. F Mallamace, HE Stanley, pp. 1–34. Amsterdam: IOS Press
- Taberlet N, Richard P, Henry E, Delannay R. 2004. The growth of a super stable heap: an experimental and numerical study. *Europhys. Lett.* 68:515–21
- Taberlet N, Richard P, Valance A, Delannay R, Losert W, et al. 2003. Super stable granular heap in a thin channel. *Phys. Rev. Lett.* 91:264301
- Veje CT, Howell DW, Behringer RP. 1999. Kinematics of a two-dimensional granular Couette experiment at the transition to shearing. *Phys. Rev. E* 59:739–45
- Volfson D, Tsimring LS, Aranson IS. 2003. Order parameter description of stationary partially fluidized shear granular flows. *Phys. Rev. Lett.* 90:254301
- Wang ZT. 2004. A note on the velocity of granular flow down a bumpy inclined plane. *Granul. Matter* 6:67–69
- Wieland JM, Gray JMNT, Hutter K. 1999. Channelized free-surface flow of cohesionless granular avalanches in a chute with shallow lateral curvature. *J. Fluid Mech.* 392:73–100
- Wood DM. 1990. *Soil Behaviour and Critical State Soil Mechanics*. Cambridge, UK: Cambridge Univ. Press



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